# Longitudinal Rescaling of Quantum Electrodynamics

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We investigate quantum longitudinal rescaling of electrodynamics, transforming coordinates as $x^{0,3} \rightarrow \lambda x^{0,3}$ and $x^{1,2} \rightarrow x^{1,2}$, to one loop. We do this by an aspherical Wilsonian renormalization, which was applied earlier to pure Yang-Mills theory. We find the anomalous powers of $\lambda$ in the renormalized couplings. Our result is only valid for $\lambda \lesssim 1$, because perturbation theory breaks down for $\lambda \ll 1$.

## I. INTRODUCTION

Field-theory actions greatly simplify under classical longitudinal rescaling, namely $x^{0,3} \rightarrow \lambda x^{0,3}$ and $x^{1,2} \rightarrow x^{1,2}$ [1][2]. Such rescaling is interesting because the center-of-mass energy transforms as $s \rightarrow \lambda^{-2} s$. Thus a large rescaling yields the high-energy limit.

We consider the effect of longitudinal rescaling of quantum electrodynamics (QED) in this paper. The Abelian gauge field with Lorentz components $A_\mu$, $\mu = 0, 1, 2, 3$, transforms as $A^{0,3} \rightarrow \lambda^{-1} A^{0,3}$ and $A^{1,2} \rightarrow A^{1,2}$. The action of the Maxwell gauge field is $S_G = -\frac{1}{4g^2} \int d^4 x F_{\mu\nu} F^{\mu\nu}$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, indices are raised with the usual Minkowski metric and $g$ is the bare electric charge. Under longitudinal rescaling, the gauge action becomes

$$S_G \longrightarrow \frac{1}{4g^2} \int d^4 x \left( F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2} F_{03}^2 - \lambda^2 F_{12}^2 \right). \tag{1.1}$$

In the high-energy limit ($\lambda \rightarrow 0$), the purely transverse terms may be neglected (this is easiest to see by an examination of the Hamiltonian). The massless Dirac action transforms as

$$S_{\text{Dirac}} \longrightarrow i \int d^4 x \, \bar\psi \left[ \lambda^{-1} \gamma^0 D_0 + \lambda^{-1} \gamma^3 D_3 + \gamma^1 D_1 + \gamma^2 D_2 \right] \psi,$$

where $D_\mu \psi = \partial_\mu \psi + i A_\mu \psi$ is the covariant derivative of $\psi$. If we make an additional rescaling of the spinor field $\psi \rightarrow \lambda^{\frac{1}{2}} \psi$ and $\bar\psi \rightarrow \lambda^{\frac{1}{2}} \bar\psi$, we obtain

$$S_{\text{Dirac}} \longrightarrow i \int d^4 x \, \bar\psi \left[ \gamma^0 D_0 + \gamma^3 D_3 + \lambda \gamma^1 D_1 + \lambda \gamma^2 D_2 \right] \psi. \tag{1.2}$$

In the quantum theory, anomalous powers of $\lambda$ will appear in the rescaled action (1.1) (1.2).

In the quantum theory, classical rescalings are no longer possible. The best-known is example is the effect of a dilatation on a classically conformal invariant field theory. Quantum corrections violate this classical symmetry.

An intuitive picture of quantum longitudinal rescaling is the following. Imagine cutting off the quantum field theory in the ultraviolet by a cubic lattice, with lattice spacing $a$. Rescaling changes the lattice spacing of longitudinal coordinates to $\lambda a$, but does not change the lattice spacing of transverse coordinates, making the cut-off anisotropic. We therefore use a two-step process, where we first integrate out high-longitudinal momentum degrees of freedom, then restore isotropy with longitudinal rescaling. Instead of a lattice, we use a sharp-momentum cut-off and Wilson's renormalization procedure, to integrate out high-momentum modes [3]. This was done in reference [4] for pure Yang-Mills theory.

In the next section we review basic Wilsonian renormalization. In Section III we examine QED with a spherical momentum cut-off (this is just to show how the procedure works). In Section IV we do the renormalization for aspherical cut-offs, which treats longitudinal momenta and transverse momenta differently. In Section V we find the quantum corrections to the QED action. We present our conclusions in the last section.

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## II. WILSONIAN RENORMALIZATION

We Wick rotate to obtain the standard Euclidean metric, so that the action is

$$S = \int d^4x \left( \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + i\bar{\psi} \slashed{D} \psi \right).$$

where raising and lowering of indices is done with the Euclidean metric and where the slash on a vector quantity $J_\mu$ is $\slashed{J} = \gamma^\mu J_\mu$, where $\gamma^\mu$ are the Euclidean Dirac matrices.

We choose cut-offs $\Lambda$ and $\tilde{\Lambda}$ to be real positive numbers with units of $cm^{-1}$ and $b$ and $\tilde{b}$ to be two dimensionless real numbers, such that $b \geq 1$ and $\tilde{b} \geq 1$. These quantities satisfy $\Lambda > \tilde{\Lambda}$ and that $\Lambda^2/b \geq \tilde{\Lambda}^2/\tilde{b}$. We introduce the ellipsoid in momentum space $\mathbb{P}$, which is the set of points $p$, such that $bp_L^2 + p_\perp^2 < \Lambda^2$. We define the smaller ellipsoid $\tilde{\mathbb{P}}$ to be the set of points $p$, such that $\tilde{b}p_L^2 + p_\perp^2 < \tilde{\Lambda}^2$. Finally, we define $\mathbb{S}$ to be the shell between the two ellipsoidal surfaces $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

We split our fields into "slow" and "fast" pieces:

$$\psi(x) = \tilde{\psi}(x) + \varphi(x), \ \bar{\psi}(x) = \bar{\tilde{\psi}}(x) + \bar{\varphi}(x), \ A_\mu(x) = \tilde{A}_\mu(x) + a_\mu(x), \tag{2.1}$$

where the Fourier components of $\psi(x)$, $\bar{\psi}(x)$ and $A_\mu(x)$ vanish outside the ellipsoid $\mathbb{P}$, the Fourier components of the slow fields $\tilde{\psi}(x)$, $\bar{\tilde{\psi}}(x)$ and $\tilde{A}_\mu(x)$ vanish outside the inner ellipsoid $\tilde{\mathbb{P}}$, and the Fourier components of the fast fields $\varphi(x)$, $\bar{\varphi}(x)$ and $a_\mu(x)$ vanish outside of the shell $\mathbb{S}$. Explicitly

$$\tilde{\psi}(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \psi(p) e^{-ip\cdot x}, \ \varphi(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} \psi(p) e^{-ip\cdot x}, \ \bar{\tilde{\psi}}(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \bar{\psi}(p) e^{ip\cdot x}, \ \bar{\varphi}(x) = \int_{\mathbb{S}} \frac{d^4p}{(2\pi)^4} \bar{\psi}(p) e^{ip\cdot x},$$

$$\tilde{A}_\mu(x) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A_\mu(p) e^{-ip\cdot x}, \ a_\mu(x) = \int_{\mathbb{S}} \frac{d^4p}{2(\pi)^4} A_\mu(p) e^{-ip\cdot x}.$$

We denote the field strength of the slow fields by $\tilde{F}_{\mu\nu} = \partial_\mu \tilde{A}_\nu - \partial_\nu \tilde{A}_\mu$. The functional integral with the ultraviolet cut-off $\Lambda$ and anisotropy parameter $b$ is

$$Z = \int_{\mathbb{P}} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A e^{-S}. \tag{2.2}$$

There is no renormalization of a gauge-fixing parameter, because we do not impose a gauge condition on the slow gauge field. We do impose a Feynman gauge condition on the fast gauge field. As we show below, counterterms must be included in the action to maintain gauge invariance. We expect that renormalizability of the the field theory implies that these have the same form at each loop order; we have not proved this, however.

Before integrating over the fast fields, we must expand the action to second order in these fields. This expansion is

$$S = \tilde{S} + S_0 + S_1 + S_2 + S_3,$$

where

$$\tilde{S} = \int d^4x (\tilde{F}_{\mu\nu} \tilde{F}^{\mu\nu} + i\bar{\tilde{\psi}} \slashed{D} \tilde{\psi}),$$

$$S_0 = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{1}{2} q^2 a_\mu(-q) a^\mu(q) + i \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \bar{\varphi}(-q) \slashed{q} \varphi(q),$$

$$S_1 = - \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \left[ \bar{\varphi}(q) \slashed{A}(p) \varphi(-q-p) \right],$$

$$S_2 = - \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \left[ \bar{\psi}(p) \slashed{a}(q) \varphi(-q-p) \right],$$

$$S_3 = - \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} \left[ \bar{\varphi}(-q-p) \slashed{a}(q) \psi(p) \right]. \tag{2.3}$$

Notice that $S_2^* = S_3$. Henceforth, we drop the tildes on the slow fields, denoting these by $\psi$, $\bar{\psi}$ and $A_\mu$, but we keep the tilde on the slow action $\tilde{S}$.

The functional integral (2.2) may be written as

$$Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A \ e^{-\tilde{S}} \int_{\mathbb{S}} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}a \ e^{-S_0} e^{-S_I} \tag{2.4}$$

where the integral of the interaction Lagranigian is $S_I = S_1 + S_2 + S_3$. To evaluate 2.4, we use the fast-field propagators

$$\langle a_\mu(p) a_\nu(q) \rangle = \frac{g^2}{q^2} g_{\mu\nu} \delta^{(4)}(p+q)(2\pi)^4 \,, \quad \langle \varphi(p) \bar{\varphi}(q) \rangle = \frac{-i\slashed{q}}{q^2} \delta^{(4)}(p+q)(2\pi)^4 \,, \tag{2.5}$$

where the brackets $\langle \rangle$ mean

$$\langle Q \rangle = \left( \int_{\mathbb{S}} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}a \, e^{-S_0} \right)^{-1} \int_{\mathbb{S}} \mathcal{D}\varphi \mathcal{D}\bar{\varphi} \mathcal{D}a \, Q \, e^{-S_0}. \tag{2.6}$$

for any quantity $Q$. We will ignore an overall free-energy renormalization from the first factor in (2.6). We use the connected-graph expansion

$$\langle e^{-S_I} \rangle = \exp[-\langle S_I \rangle + \frac{1}{2}(\langle S_I^2 \rangle - \langle S_I \rangle^2) - \frac{1}{3!}(\langle S_I^3 \rangle - 3\langle S_I^2 \rangle \langle S_I \rangle + \langle S_I \rangle^3) + \cdots], \tag{2.7}$$

for the interaction $S_I$. The terms in the exponent of (2.7) are straightforward to evaluate using (2.5). We find

$$\langle S_1 \rangle = 0$$

and

$$\frac{1}{2}\langle S_1^2 \rangle = \int_{\tilde{\mathbb{P}}} \frac{d^4 p}{(2\pi)^4} \, \Pi^{\mu\nu}(p) A_\mu(p) A_\nu(-p),$$

where the polarization tensor $\Pi^{\mu\nu}(p)$ is defined as

$$\Pi^{\mu\nu}(p) = \frac{1}{2} \int_{\mathbb{S}} \frac{d^4 q}{(2\pi)^4} \text{Tr} \left[ \frac{\slashed{q}}{q^2} \gamma^\mu \frac{\slashed{q}+\slashed{p}}{(q+p)^2} \gamma^\nu \right]. \tag{2.8}$$

Similarly

$$\langle S_2 \rangle = \langle S_3 \rangle = 0$$

and

$$\langle S_2 S_3 \rangle = \langle S_3 S_2 \rangle = \int_{\mathbb{S}} \frac{d^4 q}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4 p}{(2\pi)^4} \left[ \frac{-(\slashed{p}+\slashed{q})}{(p+q)^2} \gamma^\mu \frac{g^2}{q^2} \psi(p) \bar{\psi}(-p) \gamma_\mu \right].$$

Thus

$$\frac{1}{2}(\langle S_2 S_3 \rangle + \langle S_3 S_2 \rangle) = \int_{\tilde{\mathbb{P}}} \frac{d^4 p}{(2\pi)^4} \Sigma(p) \bar{\psi}(p) \psi(-p),$$

where the self-energy correction $\Sigma(p)$ is

$$\Sigma(p) = g^2 \int_{\mathbb{S}} \frac{d^4 q}{(2\pi)^4} \left[ \gamma^\mu \frac{i(\slashed{p}+\slashed{q})}{(p+q)^2} \gamma_\mu \frac{1}{q^2} \right] = -2g^2 \int_{\mathbb{S}} \frac{d^4 q}{(2\pi)^4} \left[ \frac{i(\slashed{p}+\slashed{q})}{q^2(q+p)^2} \right]. \tag{2.9}$$

From the cubic term in (2.7), we find

$$-\frac{1}{3!}(\langle S_I^3 \rangle - 3\langle S_I^2 \rangle \langle S_I \rangle + \langle S_I \rangle^3) = -\frac{1}{3!}\langle S_I^3 \rangle = -\langle S_1 S_2 S_3 \rangle$$

$$= \int_{\tilde{\mathbb{P}}} \frac{d^4 p}{(2\pi)^4} \int_{\tilde{\mathbb{P}}} \frac{d^4 q}{(2\pi)^4} \bar{\psi}(p) \Gamma^\mu(p,q) A_\mu(q) \psi(-p-q),$$

where the vertex correction $\Gamma^\mu(p,q)$ is

$$\Gamma^\mu(p,q) = 2g^2 \int_{\mathbb{S}} \frac{d^4 k}{(2\pi)^4} \frac{\slashed{k}\gamma^\mu(\slashed{k}+\slashed{q})}{(k-p)^2(k+q)^2 k^2}. \tag{2.10}$$

## III. SPHERICAL MOMENTUM CUT-OFFS

The cut-offs of our theory become isotropic if $b, \tilde{b} = 1$. Then the region $\mathbb{P}$ is a sphere in momentum space, whose elements $p_\mu$ satisfy $p^2 < \Lambda^2$. The region $\tilde{\mathbb{P}}$ is also a sphere, whose elements $q_\mu$ satisfy $q^2 < \tilde{\Lambda}^2$. The region $\mathbb{S}$ is a spherical shell, $\mathbb{S} = \mathbb{P} - \tilde{\mathbb{P}}$.

The polarization tensor (2.8) may be written

$$\Pi^{\mu\nu}(p) = tr \left[ \frac{1}{2} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \int_\mathbb{S} \frac{d^4 q}{(2\pi)^4} \frac{q_\alpha (q_\beta + p_\beta)}{q^2 (q+p)^2} \right].$$

We expand the integrand in powers of $p$ to second order, to find

$$\Pi^{\mu\nu}(p) = tr \left\{ \frac{1}{2} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \frac{2\pi^2}{(2\pi)^4} \left[ \int dq \frac{q^3 \delta_{\alpha\beta}}{4q^2} - \int dq \frac{q^3 p^2 \delta_{\alpha\beta}}{4q^4} \right. \right.$$
$$\left. \left. - \int dq \frac{q^3 p_\beta p^\gamma \delta_{\alpha\gamma}}{2q^4} + \int dq \frac{q^3}{6q^4} p^\gamma p^\delta (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta} + \delta_{\alpha\gamma}\delta_{\beta\delta}) \right] \right\}. \tag{3.1}$$

To obtain (3.1), we have used

$$\int_\mathbb{S} d^4 q \, q_\alpha q_\beta = 2\pi^2 \int_{\tilde{\Lambda}}^{\Lambda} dq \, q^3 \frac{\delta_{\alpha\beta} q^2}{4}$$

and

$$\int_\mathbb{S} d^4 q \, q_\alpha q_\beta q_\gamma q_\delta = \frac{1}{24} \int_\mathbb{S} d^4 q \, q^4 (\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\delta}\delta_{\gamma\beta} + \delta_{\alpha\gamma}\delta_{\beta\delta}),$$

which follow from $\mathcal{O}(4)$ symmetry. Thus the polarization tensor is

$$\Pi^{\mu\nu}(p) = tr \left\{ \frac{1}{2} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \frac{1}{8\pi^2} \left[ \frac{\delta_{\alpha\beta}}{8} (\Lambda^2 - \tilde{\Lambda}^2) - \frac{1}{12} (\delta_{\alpha\beta} p^2 + 2p_\alpha p_\beta) \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right) \right] \right\}. \tag{3.2}$$

We must remove non-gauge-invariant terms, namely those quadratic in the cut-offs, with counterterms. The remaining logarithmically-divergent part of (3.2) is

$$\hat{\Pi}^{\mu\nu}(p) = \Pi_{\mu\nu}(p) - \Pi_{\mu\nu}(0) = tr \left[ \frac{1}{2} \gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \frac{1}{8\pi^2} \left( \frac{-1}{12} \right) (\delta_{\alpha\beta} p^2 + 2p_\alpha p_\beta) \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right) \right]$$
$$= \frac{e^2}{12\pi^2} (g^{\mu\nu} p^2 - p^\mu p^\nu) \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right).$$

This gauge-invariant contribution satisfies $p^\mu \hat{\Pi}_{\mu\nu}(p) = 0$. The contribution to the action associated with this term is

$$\frac{1}{2} \langle S_1^2 \rangle = \int_{\tilde{\mathbb{P}}} \frac{d^4 p}{(2\pi)^4} \frac{1}{12\pi^2} \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right) (g^{\mu\nu} p^2 - p^\mu p^\nu) A_\mu(-p) A_\nu(p). \tag{3.3}$$

Equation (3.3) gives the effective coupling $\tilde{g}$ for the theory with cutoff $\tilde{\Lambda}$:

$$\frac{1}{4\tilde{g}^2} = \frac{1}{4g^2} + \frac{1}{12\pi^2} \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right),$$

which yields the standard result for the QED beta function:

$$\beta(\tilde{g}) = \frac{\partial \tilde{g}}{\partial \ln(\Lambda)} = \frac{g^3}{12\pi^2}.$$

The self-energy correction (2.9) is

$$\Sigma(p) = -2g^2 \int_\mathbb{S} \frac{d^4 q}{(2\pi)^4} \frac{(\slashed{q} + \slashed{p})}{q^2 (q+p)^2}.$$

We expand the integrand of $\Sigma(p)$ in powers of $p$, which gives

$$\Sigma(p) = -2g^2 \frac{\gamma^\alpha}{8\pi^2} \int dq \left[ \frac{-p_\alpha}{2q} + \frac{p_\alpha}{q} \right] = -\frac{g^2 \slashed{p}}{8\pi^2} \ln \left( \frac{\Lambda}{\tilde{\Lambda}} \right). \tag{3.4}$$

The vertex correction (2.10) is

$$\Gamma^\mu(p,q) = 2g^2 \gamma^\alpha \gamma^\mu \gamma^\beta \int_{\mathbb{S}} \frac{d^4k}{(2\pi)^4} \frac{k_\alpha(k_\beta + q_\beta)}{(k-p)^2(k+q)^2 k^2}.$$

Expanding the integrand in powers of $p$,

$$\Gamma^\mu(p,q) = 2g^2 \gamma^\alpha \gamma^\mu \gamma^\beta \int_{\mathbb{S}} \frac{d^4k}{(2\pi)^4} \left[ \frac{\delta_{\alpha\beta}}{4k^4} - \frac{\delta_{\alpha\beta} p^2}{4k^6} - \frac{\delta_{\alpha\beta} q^2}{4k^6} + \frac{\delta_{\alpha\gamma}}{2k^6}(q_\beta p^\gamma - q_\beta q^\gamma) \right.$$
$$\left. - \frac{\delta_{\alpha\beta}\delta_{\gamma\delta} + \delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\gamma}}{6k^6}(q^\gamma p^\delta - q^\gamma q^\delta - p^\gamma p^\delta) \right].$$

We retain only the divergent part of $\Gamma^\mu(p,q)$, namely the first term:

$$\Gamma^\mu(p,q) = 2g^2 \frac{\gamma^\alpha \gamma^\mu \gamma^\beta}{8\pi^2} \frac{\delta_{\alpha\beta}}{4} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) = \frac{-g^2 \gamma^\mu}{8\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right). \tag{3.5}$$

## IV.   ELLIPSOIDAL CUT-OFFS

Next we consider the more general ellipsoidal case. The integration over $\mathbb{S}$ is done by changing variables from $q_\mu$ to two variables $u$ and $w$, with units of momentum squared, and two angles $\theta$ and $\phi$. These variables are defined by

$$q_1 = \sqrt{u} \cos\theta, \; q_2 = \sqrt{u} \sin\theta, \; q_3 = \sqrt{w-u} \cos\phi, \; q_0 = \sqrt{w-u} \sin\phi.$$

The integration over these variables is

$$\int_{\mathbb{S}} d^4q = \frac{1}{4} \int_0^{2\pi} d\theta \int_0^{2\pi} d\phi \left[ \int_0^{\tilde{\Lambda}^2} du \int_{\tilde{b}^{-1}\tilde{\Lambda}^2 + (1-\tilde{b}^{-1})u}^{b^{-1}\Lambda^2 + (1-b^{-1})u} dw + \int_{\tilde{\Lambda}^2}^{\Lambda^2} du \int_u^{b^{-1}\Lambda^2 + (1-b^{-1})u} dw \right].$$

We have a $\mathcal{O}(2) \times \mathcal{O}(2)$ symmetry, generated by the translations $\theta \to \theta + d\theta$ and $\phi \to \phi + d\phi$, rather than $\mathcal{O}(4)$ symmetry.

Our three corrections are expressed in terms of the integrals

$$A_{\alpha\beta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{q_\alpha q_\beta}{q^4},$$
$$B_{\alpha\beta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{q_\alpha q_\beta}{q^6},$$
$$C_{\alpha\beta\gamma\delta} = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{q_\alpha q_\beta q_\gamma q_\delta}{q^8}$$

and

$$D = \int_{\mathbb{S}} \frac{d^4q}{(2\pi)^4} \frac{1}{q^4}. \tag{4.1}$$

By inspection we write

$$\Pi^{\mu\nu}(p) = tr\left[\frac{1}{2}\gamma^\mu \gamma^\alpha \gamma^\nu \gamma^\beta \left[A_{\alpha\beta} + 4C_{\alpha\beta\gamma\delta}p^\gamma p^\delta - p^2 B_{\alpha\beta} - 2B_{\alpha\gamma}p_\beta p^\gamma\right]\right],$$
$$\Sigma(p) = -2g^2 \gamma^\alpha \left[-2B_{\alpha\gamma}p^\gamma + p_\alpha D\right]$$

and

$$\Gamma^\mu = 2g^2 \gamma^\alpha \gamma^\mu \gamma^\beta B_{\alpha\beta}.$$

We use $C$ and $D$ to denote Lorentz indices taking only the values 1 and 2. We use $\Omega$ and $\Xi$ to denote Lorentz indices taking

only the values 3 and 0. The integration is straightforward, though tedious. We present only the results:

$$A_{CD} = \frac{\delta_{CD}}{32\pi^2}\Lambda^2\left[1 + \frac{b}{(b-1)^2}(1-b+\ln b)\right] - \frac{\delta_{CD}}{32\pi^2}\tilde{\Lambda}^2\left[1 + \frac{\tilde{b}}{(\tilde{b}-1)^2}(1-\tilde{b}+\ln\tilde{b})\right],$$

$$A_{\Omega\Xi} = \frac{\delta_{\Omega\Xi}}{32\pi^2}\left[\Lambda^2\left(\frac{1}{b-1} - \frac{\ln b}{(b-1)^2}\right) - \tilde{\Lambda}^2\left(\frac{1}{\tilde{b}-1} - \frac{\ln\tilde{b}}{(\tilde{b}-1)^2}\right)\right],$$

$$A_{C\Omega} = 0,$$

$$B_{CD} = \frac{\delta_{CD}}{32\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{\delta_{CD}}{64\pi^2}\left[\frac{b^2\ln b}{(b-1)^2} - \frac{b}{b-1}\right] + \frac{\delta_{CD}}{64\pi^2}\left[\frac{\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^2} - \frac{\tilde{b}}{\tilde{b}-1}\right],$$

$$B_{\Omega\Xi} = \frac{\delta_{\Omega\Xi}}{32\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{b(b-2)\ln b}{(b-1)^2} + \frac{b}{b-1}\right] + \frac{\delta_{\Omega\Xi}}{64\pi^2}\left[\frac{\tilde{b}(\tilde{b}-2)\ln\tilde{b}}{(\tilde{b}-1)^2} + \frac{\tilde{b}}{\tilde{b}-1}\right],$$

$$B_{C\Omega} = 0,$$

$$C_{CCCC} = \frac{1}{64\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{128\pi^2}\frac{b^3}{(b-1)^3}\left(\ln b - \frac{2(b-1)}{b} + \frac{(b-1)(b+1)}{2b^2}\right)$$
$$+\frac{1}{128\pi^2}\frac{\tilde{b}^3}{(\tilde{b}-1)^3}\left(\ln\tilde{b} - \frac{2(\tilde{b}-1)}{\tilde{b}} + \frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}}\right),$$

$$C_{1122} = \frac{C_{CCCC}}{3},$$

$$C_{\Omega\Omega\Omega\Omega} = \frac{1}{64\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{128\pi^2}\left[\frac{b^3}{(b-1)^3}\left(\ln b - \frac{2(b-1)}{b} + \frac{(b-1)(b+1)}{2b^2}\right) + \frac{3b\ln b}{b-1} - \frac{3b^2\ln b}{(b-1)^2} + \frac{3b}{b-1}\right]$$
$$+\frac{1}{128\pi^2}\left[\frac{\tilde{b}^3}{(\tilde{b}-1)^3}\left(\ln\tilde{b} - \frac{2(\tilde{b}-1)}{\tilde{b}} + \frac{(\tilde{b}-1)(\tilde{b}+1)}{2\tilde{b}^2}\right) + \frac{3\tilde{b}\ln\tilde{b}}{\tilde{b}-1} - \frac{3\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^2} + \frac{3\tilde{b}}{\tilde{b}-1}\right],$$

$$C_{0033} = \frac{C_{\Omega\Omega\Omega\Omega}}{3},$$

$$C_{CC\Omega\Omega} = \frac{1}{192\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{384\pi^2}\left[\frac{-2b^2\ln b}{(b-1)^3} + \frac{b^2\ln b + 2b}{(b-1)^2}\right] + \frac{1}{384\pi^2}\left[\frac{-2\tilde{b}^2\ln\tilde{b}}{(\tilde{b}-1)^3} + \frac{\tilde{b}^2\ln\tilde{b} + 2\tilde{b}}{(\tilde{b}-1)^2}\right],$$

$$D = \frac{1}{8}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{1}{16\pi^2}\left[\frac{b\ln b}{b-1} - \frac{\tilde{b}\ln\tilde{b}}{\tilde{b}-1}\right]. \tag{4.2}$$

Setting $b = \tilde{b}$ in (4.2), we recover the results from the spherical integration done in Section III. We simplify by setting $b = 1$ and $\tilde{b} \approx 1$, using the expansion $\tilde{b} = 1 + \ln\tilde{b} + \frac{\ln^2\tilde{b}}{2} + \frac{\ln^3\tilde{b}}{3!} + \cdots$ and $\ln\tilde{b} = \ln\tilde{b} - \frac{\ln^2\tilde{b}}{2} + \frac{\ln^3\tilde{b}}{3} - \frac{\ln^4\tilde{b}}{4} + \cdots$, dropping terms of second order in $\ln\tilde{b}$.

The vertex correction is

$$\Gamma^\mu(p,q) = 2g^2\gamma^\alpha\gamma^\mu\gamma^\beta B_{\alpha\beta} = 2g^2\left(\frac{-\gamma^\mu}{16\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) - \frac{\gamma^\mu}{16\pi^2}\ln\tilde{b} + \frac{g^{C\mu}\gamma_C}{32\pi^2}\frac{5}{6}\ln\tilde{b} + \frac{g^{\Omega\mu}\gamma_\Omega}{32\pi^2}\frac{7}{6}\ln\tilde{b}\right). \tag{4.3}$$

The self-energy correction is

$$\Sigma(p) = -2g^2\gamma^\alpha[-2B_{\alpha\beta}p^\beta + p_\alpha D] = -2g^2\left[\frac{\gamma^\mu p_\mu}{16\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^C p_C}{6} - \frac{1}{32\pi^2}\ln\tilde{b}\frac{\gamma^\Omega p_\Omega}{6}\right]. \tag{4.4}$$

In the next section, we show how these affect the effective action.

The most general gauge-field action which is quadratic in $A_\mu$, is $\mathcal{O}(2) \times \mathcal{O}(2)$ invariant and gauge invariant, to leading order is

$$S_{\text{quadratic}} = \int_\mathbb{P}\frac{d^4p}{(2\pi)^4}A(-p)^T\left[a_1 M_1(p) + a_2 M_2(p) + a_3 M_3(p)\right]A(p),$$

where

$$M_1(p) = \begin{pmatrix} p_2^2 & -p_1p_2 & 0 & 0 \\ -p_1p_2 & p_1^2 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, M_2(p) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & p_0^2 & -p_3p_0 \\ 0 & 0 & -p_3p_0 & p_3^2 \end{pmatrix},$$

$$M_3(p) = \begin{pmatrix} p_L^2 & 0 & -p_1p_3 & -p_1p_0 \\ 0 & p_L^2 & -p_2p_3 & -p_2p_0 \\ -p_1p_3 & -p_2p_3 & p_\perp^2 & 0 \\ -p_1p_0 & -p_2p_0 & 0 & p_\perp^2 \end{pmatrix},$$

and $a_1$, $a_2$ and $a_3$ are real numbers. Any part of the polarization tensor that cannot be expressed in terms of these matrices (i.e. $\int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^4} A_\mu(-p)\Pi_{\mu\nu}(p)A_\nu(p) - S_{\text{quadratic}}$) must be removed with counterterms. After some work we find

$$\Pi^{\mu\nu}(p) = tr\left[\tfrac{1}{2}\gamma^\mu\gamma^\alpha\gamma^\nu\gamma^\beta\left[A_{\alpha\beta} + 4C_{\alpha\beta\gamma\delta}p^\gamma p^\delta - p^2 B_{\alpha\beta} - 2B_{\alpha\gamma}p_\beta p^\gamma\right]\right]$$

$$= \tfrac{1}{12\pi^2}\ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right)(p^2\mathbf{1} - pp^T)^{\mu\nu} + \tfrac{5\ln\tilde{b}}{48\pi^2}(p^2\mathbf{1} - pp^T)^{\mu\nu}$$

$$+\tfrac{\ln\tilde{b}}{128\pi^2}\left[\tfrac{8}{9}M_3 + \tfrac{40}{9}M_2 - \tfrac{104}{9}M_1 + \tfrac{8}{3}\left(\begin{array}{cc} \left(\tfrac{17}{6}p_\perp^2 + \tfrac{4}{3}p_L^2\right)\mathbf{1}_{2\times 2} & 0 \\ 0 & -\left(\tfrac{7}{6}p_L^2 + \tfrac{14}{3}p_\perp^2\right)\mathbf{1}_{2\times 2}\end{array}\right)\right]^{\mu\nu}. \tag{4.5}$$

This determines $a_1$, $a_2$ and $a_3$, so that

$$S_{\text{diff}} = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi^2)} A(-p)^T M_{\text{diff}} A(p) = \int_{\tilde{\mathbb{P}}} \frac{d^4p}{(2\pi)^2} A(-p)^T \Pi A(p) - S_{\text{quadratic}} \tag{4.6}$$

is maximally non-gauge invariant. The matrix $M_{\text{diff}}$ is the last diagonal matrix in (4.5). The quantity $S_{\text{diff}}$ is proportional to the local counterterms to include in the action. We find

$$a_1 = \frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} - \frac{1}{128\pi^2}\frac{104}{9}\right)\ln\tilde{b}, \ a_2 = \frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{40}{9}\right)\ln\tilde{b},$$

and

$$a_3 = \frac{1}{12\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \left(\frac{5}{48\pi^2} + \frac{1}{128\pi^2}\frac{8}{9}\right)\ln\tilde{b}$$

In the next section, we show how the action changes under renormalization. We then rescale to restore the isotropy.

## V. THE RESCALED EFFECTIVE ACTION

We define the effective action $S'$, which contains the effects of integrating out the fast fields, by

$$Z = \int_{\tilde{\mathbb{P}}} \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, e^{-S'} = \int_{\mathbb{P}} \mathcal{D}\psi\mathcal{D}\bar{\psi}\mathcal{D}A\, e^{-\tilde{S}} \int_{\mathbb{S}} \mathcal{D}\varphi\mathcal{D}\bar{\varphi}\mathcal{D}a\, e^{-S_0}e^{-R},$$

where $S' = \int d^4x\left[\mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{vertex}} + \mathcal{L}_{\text{gauge}}\right] = \int d^4x\left[\mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{gauge}}\right]$. To one loop

$$\mathcal{L}_{\text{Fermion}} = \bar{\psi}i(\slashed{\partial} - \Sigma(\partial))\psi,$$
$$\mathcal{L}_{\text{vertex}} = \bar{\psi}(\gamma^\mu - \Gamma^\mu)A_\mu\psi$$

and

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} + A_\mu(\sum_{i=1}^{3} a_i M_i^{\mu\nu}(\partial))A_\nu.$$

Explicitly, $\mathcal{L}_{\text{vertex}}$ is

$$\mathcal{L}_{\text{vertex}} = \bar{\psi}\left[\gamma^C\left(1 + \tfrac{g^2}{8\pi^2}\ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{g^2}{8\pi^2}\ln\tilde{b} - \tfrac{5g^2}{96\pi^2}\ln\tilde{b}\right)A_C\right.$$

$$+\gamma^\Omega\left(1 + \tfrac{g^2}{8\pi^2}\ln\left(\tfrac{\Lambda}{\tilde{\Lambda}}\right) + \tfrac{g^2}{8\pi^2}\ln\tilde{b} - \tfrac{7g^2}{96\pi^2}\ln\tilde{b}\right)A_\Omega\Big]\psi$$

$$= R\bar{\psi}\left[\gamma^C A_C + \lambda^{\frac{g^2}{24\pi^2 R}}\gamma^\Omega A_\Omega\right]\psi,$$

where

$$R = \tilde{R} + \left(\frac{g^2}{8\pi^2} - \frac{5g^2}{96\pi^2}\right)\ln\tilde{b} \approx \tilde{R}\tilde{b}^{\frac{7g^2}{96\pi^2 \tilde{R}}} = \tilde{R}\lambda^{-\frac{7g^2}{48\pi^2 \tilde{R}}},$$

and

$$\tilde{R} = 1 + \frac{g^2}{8\pi^2}\ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right),$$

for small $\ln \tilde{b}$, where we have identified $\tilde{b} = \lambda^{-2}$.

The term $\mathcal{L}_{\text{Fermion}}$, which contains the self-energy correction:

$$\mathcal{L}_{\text{Fermion}} = \bar{\psi}i \left[ \gamma^C \partial_C \left( 1 + \frac{g^2}{8\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{g^2}{8\pi^2} \ln \tilde{b} - \frac{5g^2}{96\pi^2} \ln \tilde{b} - \frac{g^2}{16\pi^2} \ln \tilde{b} \right) \right.$$
$$\left. + \gamma^\Omega \partial_\Omega \left( 1 + \frac{g^2}{8\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{g^2}{8\pi^2} \ln \tilde{b} - \frac{5g^2}{96\pi^2} \ln \tilde{b} - \frac{g^2}{12\pi^2} \ln \tilde{b} \right) \right] \psi$$
$$= R' \bar{\psi}i \left[ \gamma^C \partial_C + \lambda^{\frac{g^2}{24\pi^2 R}} \gamma^\Omega \partial_\Omega \right] \psi,$$

where

$$R' = R\tilde{b}^{-\frac{g^2}{16\pi^2 R}} = R\lambda^{\frac{g^2}{8\pi^2 R}}.$$

For consistency, we write $\mathcal{L}_{\text{vertex}}$ in terms of $R'$,

$$\mathcal{L}_{\text{vertex}} = R'\lambda^{-\frac{g^2}{8\pi^2 R}} \bar{\psi} \left[ \gamma^C A_C + \lambda^{\frac{g^2}{24\pi^2 R}} \gamma^\Omega A_\Omega \right] \psi.$$

We must rescale the gauge field by

$$\lambda^{\frac{-g^2}{8\pi^2 R}} A_\mu \to A_\mu, \tag{5.1}$$

to express $\mathcal{L}_{\text{Dirac}} = \mathcal{L}_{\text{Fermion}} + \mathcal{L}_{\text{vertex}}$ in terms of a covariant derivative. This rescaling also affects $\mathcal{L}_{\text{gauge}}$. We now have

$$\mathcal{L}_{\text{Dirac}} = R' \bar{\psi}i \left[ \gamma^C D_C + \lambda^{\frac{g^2}{24\pi^2 R}} \gamma^\Omega D_\Omega \right] \psi.$$

Rescaling the spinor field by

$$R'\lambda^{-1+\frac{g^2}{24\pi^2 R}} \bar{\psi}\psi \to \bar{\psi}\psi,$$

gives us the form

$$\mathcal{L}_{\text{Dirac}} = \bar{\psi}i \left[ \lambda^{1-\frac{g^2}{24\pi^2 R}} \gamma^C D_C + \gamma^\Omega D_\Omega \right] \psi. \tag{5.2}$$

Including vacuum-polarization corrections, $\mathcal{L}_{\text{gauge}}$ becomes

$$\mathcal{L}_{\text{gauge}} = \left( \frac{1}{4g^2} + \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{1}{9\pi^2} \ln \tilde{b} \right) (F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2)$$
$$+ \left( \frac{1}{4g^2} + \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{1}{9\pi^2} \ln \tilde{b} + \frac{1}{36\pi^2} \ln \tilde{b} \right) F_{03}^2$$
$$+ \left( \frac{1}{4g^2} + \frac{1}{12\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right) + \frac{1}{9\pi^2} \ln \tilde{b} - \frac{7}{72\pi^2} \ln \tilde{b} \right) F_{12}^2.$$

We introduce the effective coupling $g_{\text{eff}}$,

$$\frac{1}{g_{\text{eff}}^2} = \frac{1}{\tilde{g}^2} + \frac{4}{9\pi^2} \ln \tilde{b} \approx \frac{1}{\tilde{g}^2} \tilde{b}^{\frac{4\tilde{g}^2}{9\pi^2}} = \frac{1}{\tilde{g}^2} \lambda^{\frac{-8}{9\pi^2}\tilde{g}^2},$$

where

$$\frac{1}{\tilde{g}^2} = \frac{1}{g^2} + \frac{1}{3\pi^2} \ln\left(\frac{\Lambda}{\tilde{\Lambda}}\right). \tag{5.3}$$

Then

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g_{\text{eff}}^2} \left( F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2 + \lambda^{-\frac{2}{9\pi^2}\tilde{g}^2} F_{03}^2 + \lambda^{\frac{7}{9\pi^2}\tilde{g}^2} F_{12}^2 \right).$$

We finally rescale the gauge field with the factor from (5.1),

$$F_{\mu\nu}^2 \rightarrow \lambda^{\frac{g^2}{4\pi^2\bar{R}}} F_{\mu\nu}^2,$$

and define a new effective coupling $g'_{\text{eff}}$ that absorbs this factor

$$\frac{1}{g'^2_{\text{eff}}} = \frac{1}{\tilde{g}^2} \lambda^{-\frac{8}{9\pi^2}\tilde{g}^2 + \frac{g^2}{4\pi^2\bar{R}}}, \tag{5.4}$$

$$\mathcal{L}_{\text{gauge}} = \frac{1}{4g'^2_{\text{eff}}} \left( F_{01}^2 + F_{02}^2 + F_{13}^2 + F_{23}^2 + \lambda^{-\frac{2}{9\pi^2}\tilde{g}^2} F_{03}^2 + \lambda^{\frac{7}{9\pi^2}\tilde{g}^2} F_{12}^2 \right). \tag{5.5}$$

Our final result, after longitudinal rescaling and Wick-rotating back to Minkowski space-time is

$$\mathcal{L} = \mathcal{L}_{\text{Dirac}} + \mathcal{L}_{\text{gauge}} = \bar{\psi} i \left[ \lambda^{1 - \frac{g^2}{12\pi^2\bar{R}}} \gamma^C D_C + \gamma^\Omega D_\Omega \right] \psi$$
$$+ \frac{1}{4g'^2_{\text{eff}}} \left( F_{01}^2 + F_{02}^2 - F_{13}^2 - F_{23}^2 + \lambda^{-2 - \frac{2}{9\pi^2}\tilde{g}^2} F_{03}^2 - \lambda^{2 + \frac{7}{9\pi^2}\tilde{g}^2} F_{12}^2 \right). \tag{5.6}$$

## VI. CONCLUSIONS

In comparing our final action (5.6) to its classically-rescaled counterpart ((1.1) and (1.2)) we notice anomalous powers of $\lambda$, as well as corrections to the coupling constant (5.4). As we take $\lambda \rightarrow 0$, the effective coupling $g'_{\text{eff}}$ becomes weaker, as opposed to the growing coupling constant of QCD [4]. The high-energy limit presents a problem for the effective transverse electric charge, which describes how transverse components of the gauge field are coupled to the Dirac field. The transverse electric charge $e_\perp$ is given by the coefficient of $F_{12}^2$ in the action, which we have called $\frac{1}{4e_\perp^2}$, and is

$$e_\perp = \tilde{g}\lambda^{-1 + \frac{\tilde{g}^2}{18\pi^2} - \frac{g^2}{8\pi^2\bar{R}}}. \tag{6.1}$$

This coupling diverges even in the classical theory (note the factor $\lambda^{-1}$ in (6.1)). The divergence is enhanced by the quantum correction. Our final action (5.6) is strictly valid only for $\lambda \lesssim 1$. Naively substituting $\lambda \ll 1$ in our result suggest what the real high-energy action should be, though we cannot tell if this form is truly correct. We can extrapolate our results to higher energies, provided our couplings (particularly $e_\perp$) do not become large enough to invalidate perturbation theory. Naively, a Landau pole is encountered by solving our renormalization grouo equations at sufficiently high energy (the Landau pole itself is not meaningful, but is just a signal that extrapolating to arbitrarily high energies is impossible).

It is interesting to see the role that fermions play in longitudinal rescaling. Pure Yang-Mills theory has been studied in Reference [4], where anomalous powers of the rescaling factor $\lambda$ were found. The quantum corrections due to fermions have the opposite effect to corrections due to the non-Abelian gauge fields (the anomalous powers of $\lambda$ that arise have opposite signs). We intend to study how QCD behaves with the inclusion of quarks. Particularly, we wish to see how anomalous dimensions depend on the number of flavors.

Eventually we wish to study longitudinal rescaling with a manifestly gauge invariant, but anisotropic, regularization (such as some version of dimensional regularization). It may then be possible to use the background-field method instead of Wilsonian renormalization.

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